訊號處理
Signal Processing

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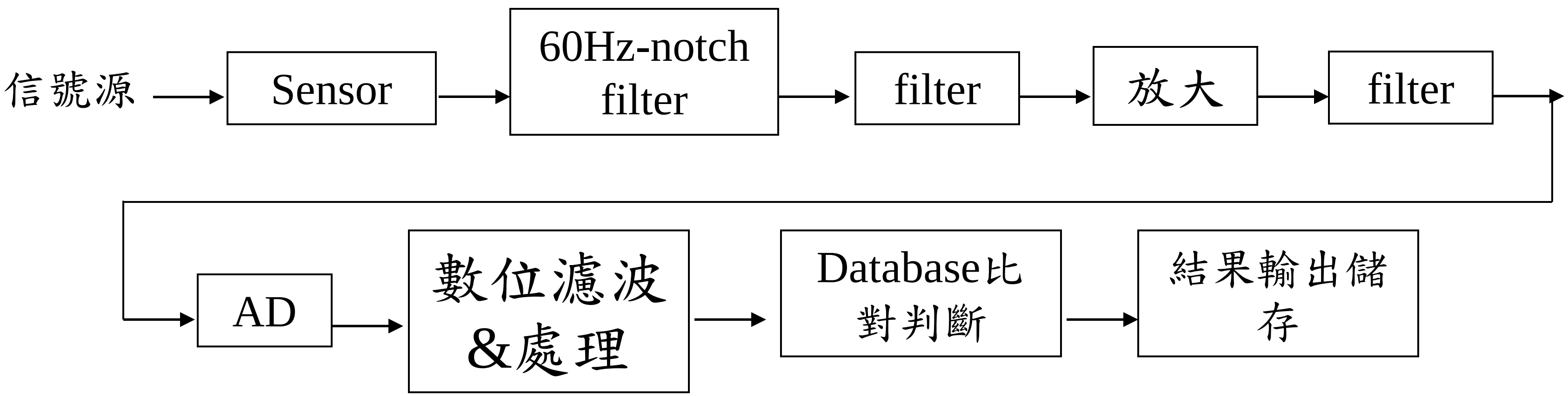
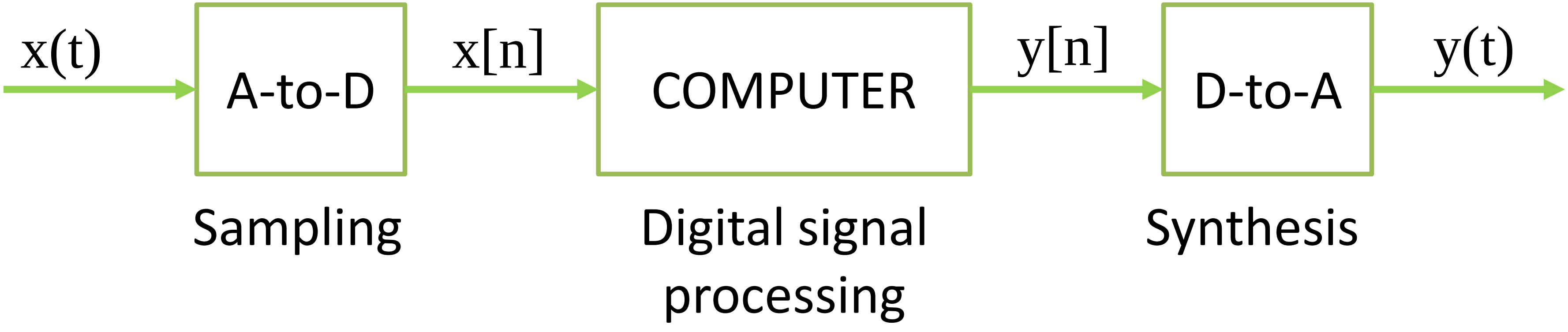
硬體常用方塊圖



- 生醫信號大小微弱，S/N ratio低
  - 須進行濾波以及訊號放大
  - 放大之後， 再轉換成數位訊號進一步處理

（生醫）訊號的特性

- Analog vs. Digital
  - Continuous vs. Discrete
  - Electronic devices vs. Microprocessor
- 生理訊號 vs. 訊號處理



生醫訊號處理

- 終極目標：從量測資料萃取出有用的特定生物資訊
  - 消除雜訊、增強訊號、濾波
  - 特徵擷取 (feature extraction)
  - 圖樣辨識 (pattern recognition)
  - 診斷用分類
  - ……不可能通通講

這次的主題

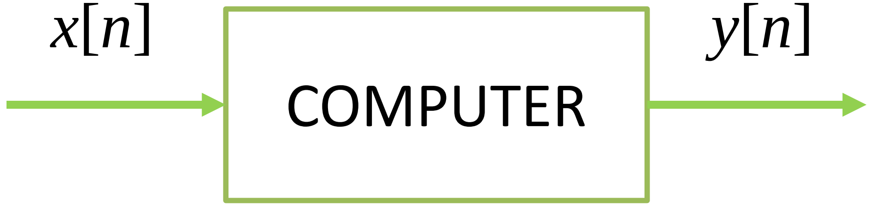
- 基本概念
  - Fourier transform, Spectrum
- 取樣
  - Sampling theorem
- 簡單濾波
  - Filtering: FIR and IIR

數位訊號處理

$x[n]$ $y[n]$

$a \cdot x_1[n] + b \cdot x_2[n]$ → $a \cdot y_1[n] + b \cdot y_2[n]$

$x[n-n_0]$ → $y[n-n_0]$

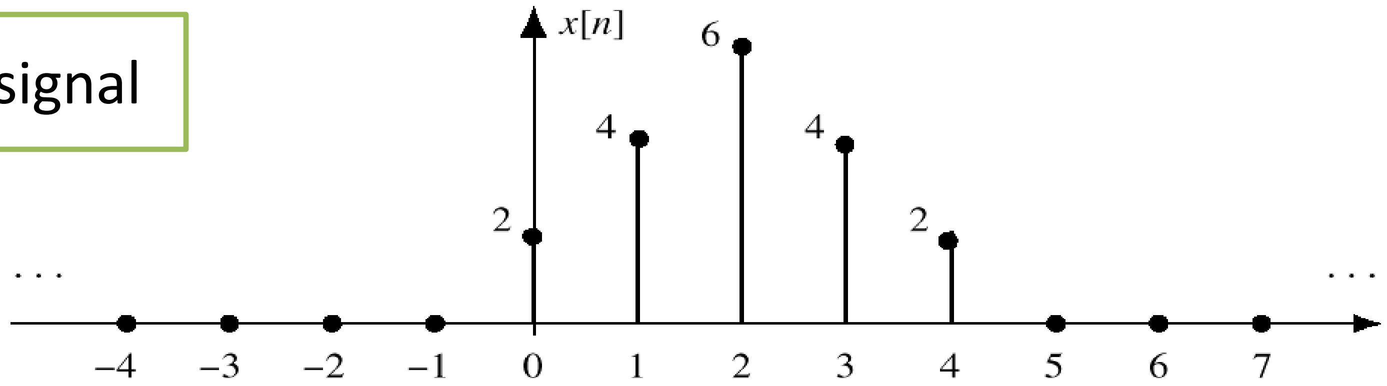


Linear and time-invariant (LTI) system

- 這堂課的系統均為 linear and time-invariant
  - Linear ??
  - Time-invariant ??
- Operate on x[n] to derive y[n]

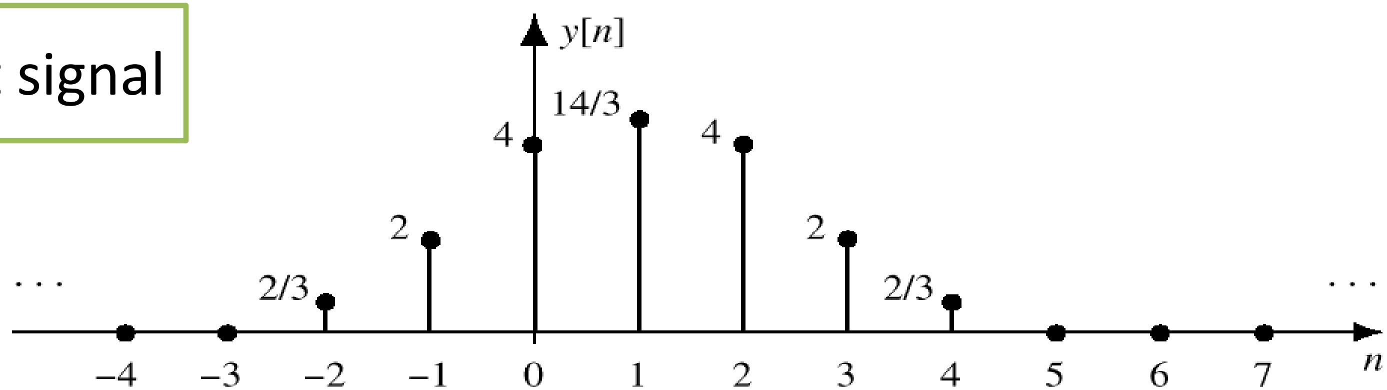
Example: 3-point averaging filter

Input signal



$$y[n] = \frac{1}{3}(x[n] + x[n+1] + x[n+2])$$

Output signal



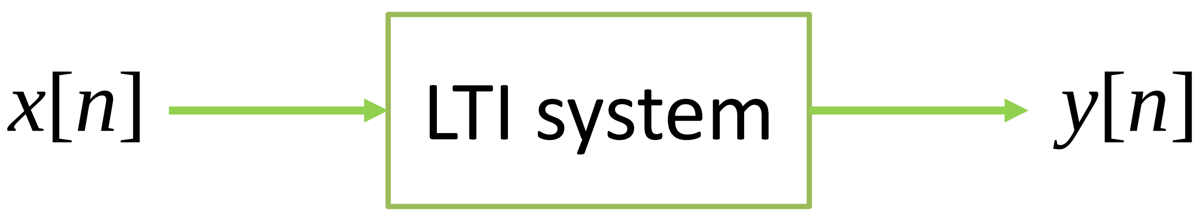
數位訊號處理

- 運用加減乘除達成訊號處理
  - 不需要電阻、電容
- 運算彈性大
  - 調整參數馬上得到想要的效果
  - 而且可以事後再處理

以數學式來表示訊號處理

- Impulse function (delta function, $\delta(t)$)
- Convolution ($f * g$)

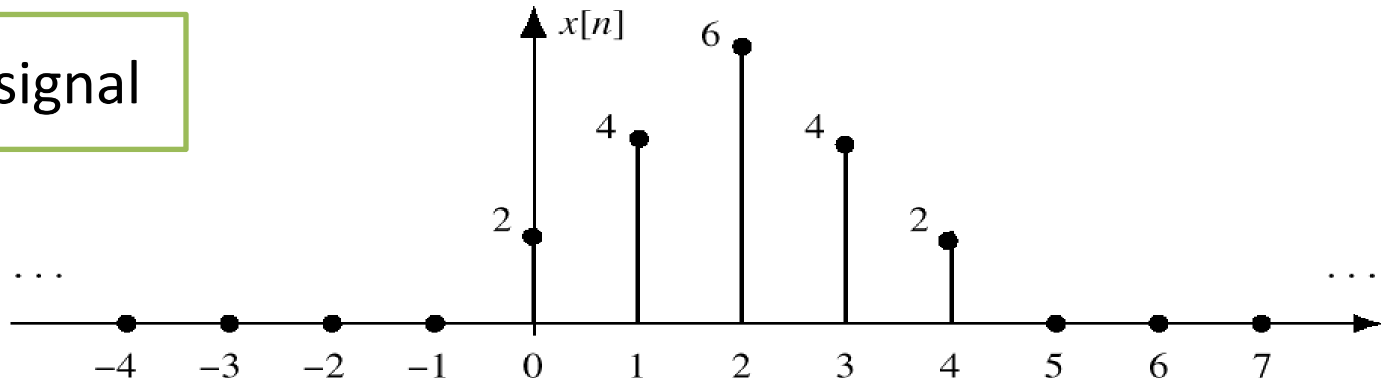
Impulse response: $h[n]$



$$y[n] = h * x[n]$$

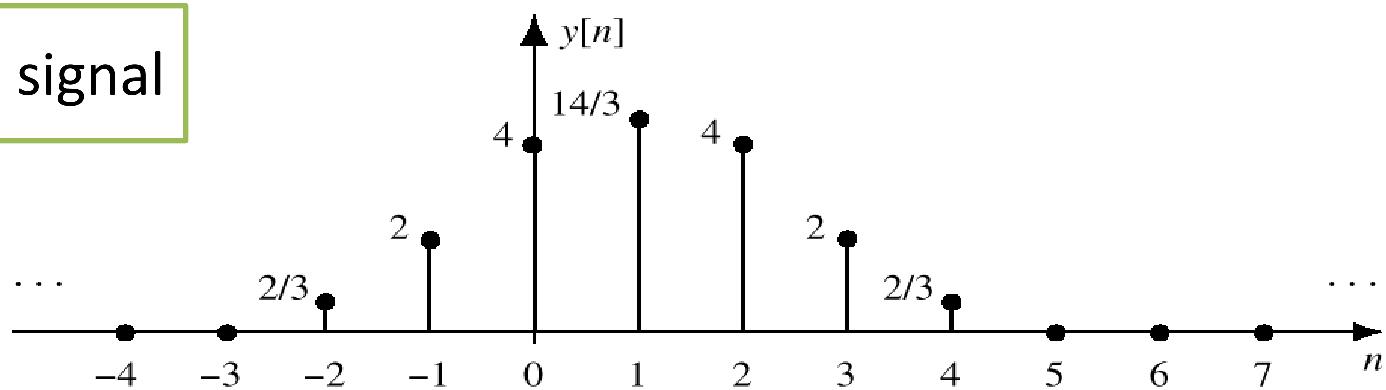
What is its impulse response?

Input signal



$$y[n] = \frac{1}{3}(x[n] + x[n+1] + x[n+2])$$

Output signal



Fourier transform

- 描述時域與頻譜的關係
- 時域的訊號可視為許多不同頻率的正弦波之線性組合

Fourier transform

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$x(t) \xrightarrow{F} X(\omega)$$

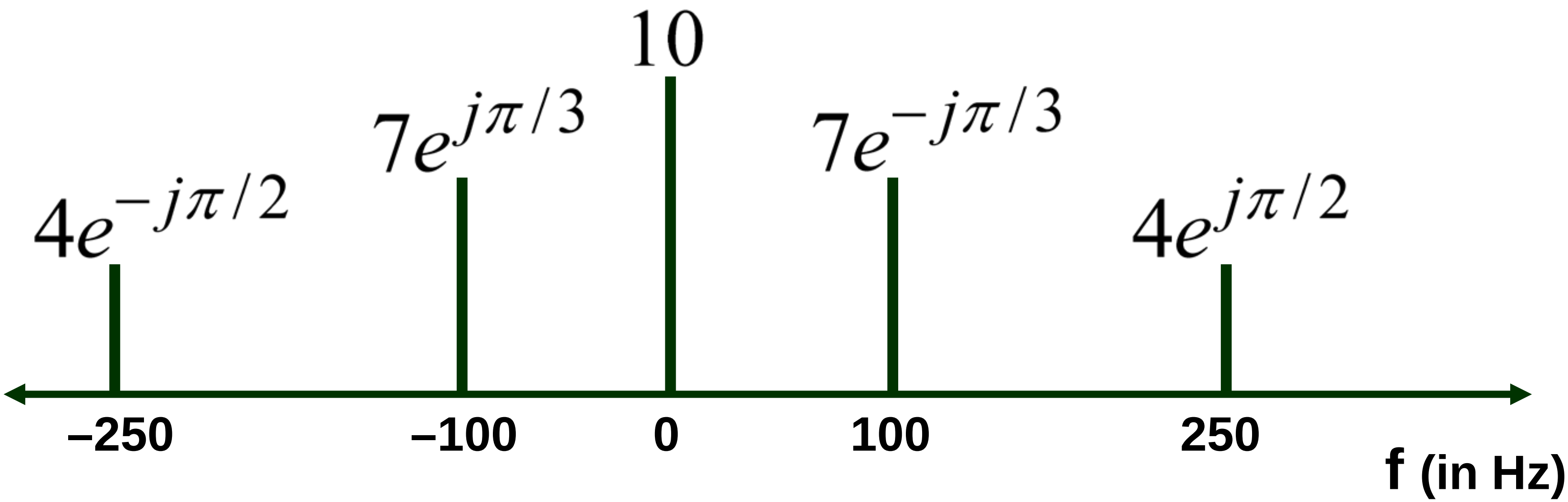
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) \xrightarrow{F^{-1}} x(t)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega \quad , \quad \omega = 2\pi f$$

一樣來看個例子

Given a spectrum, $x(t)$ = ?

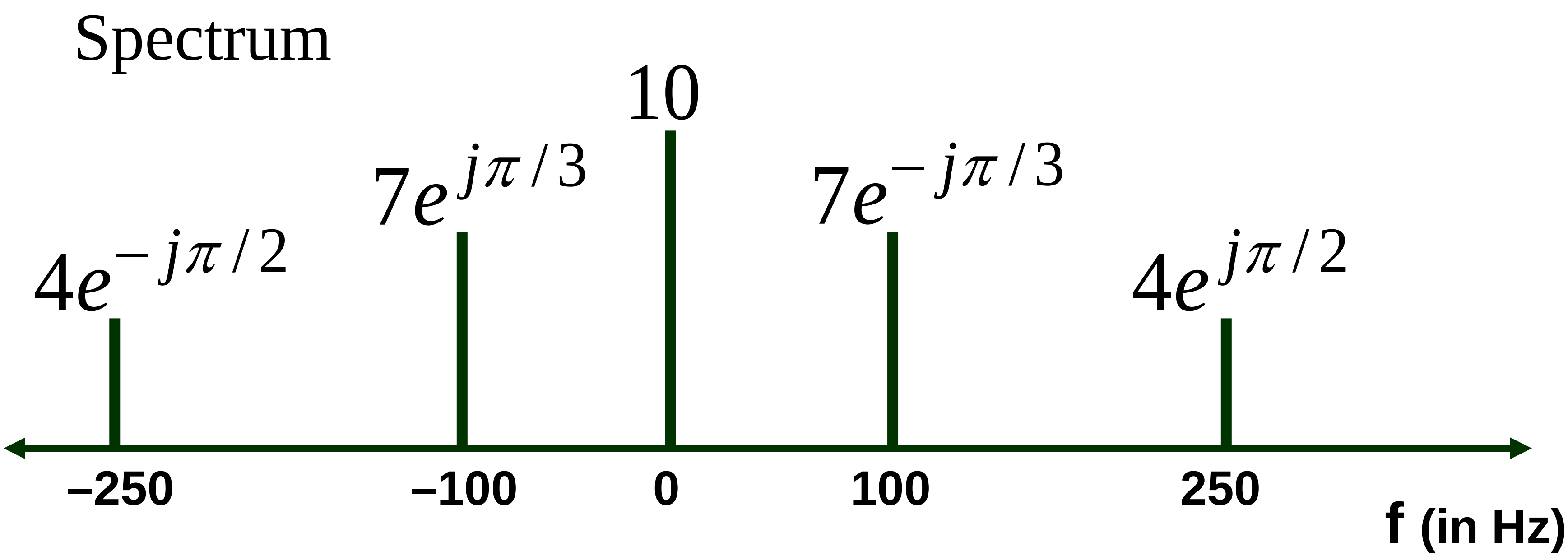


$$x(t) = 10e^{j2\pi(0)t} +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

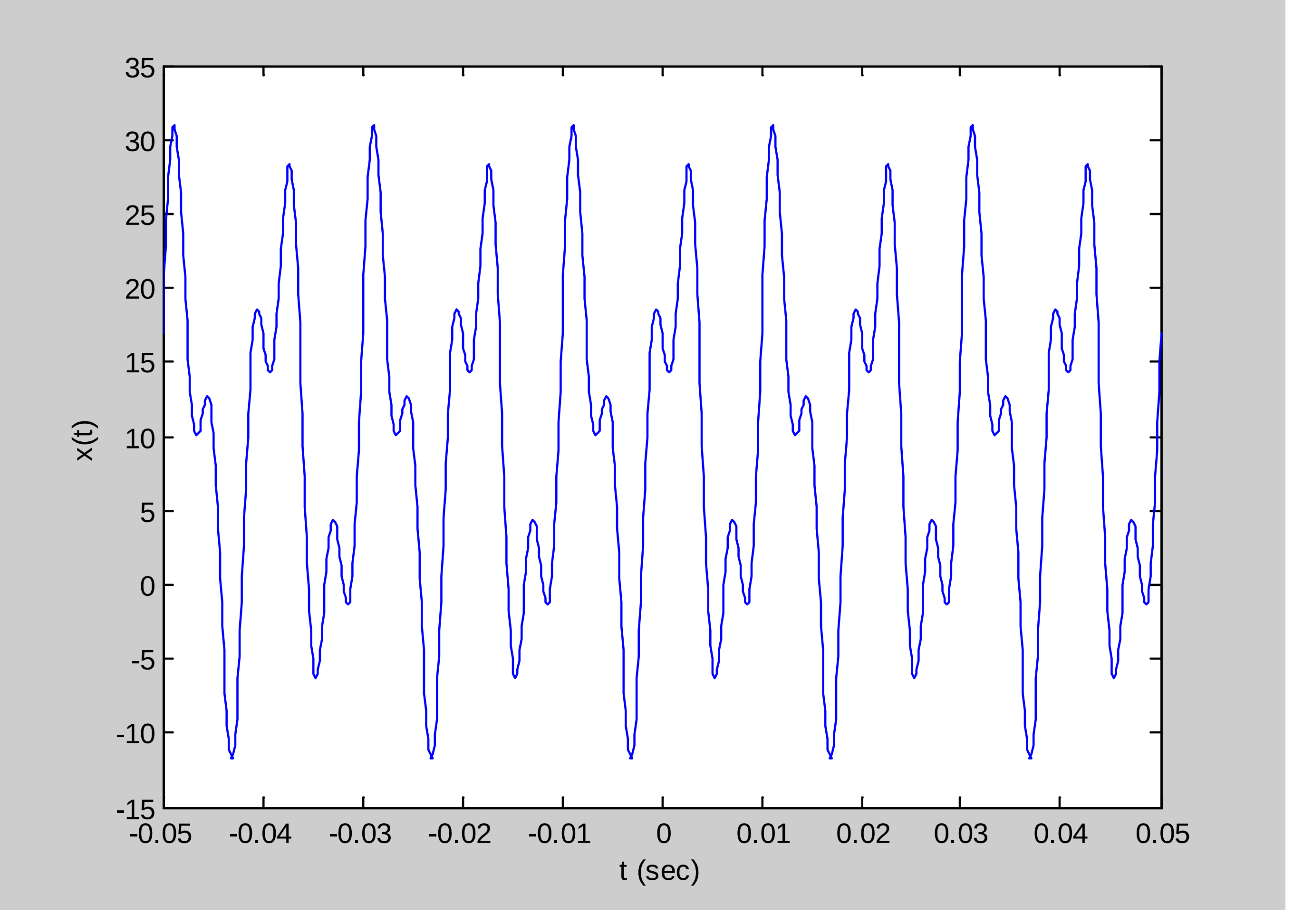
$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

所以在時域上的表示…



$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

From spectrum to signal



How about from signal to spectrum?

Properties of FT

$$ax(t) + by(t) \longleftrightarrow aX(\omega) + bY(\omega)$$

$$x(-t) \longleftrightarrow X(-\omega)$$

$$x(at) \longleftrightarrow \frac{1}{|a|} X(\frac{\omega}{a})$$

$$x(t) * y(t) \longleftrightarrow X(\omega)Y(\omega)$$

$$x(t)y(t) \longleftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega)$$

$$\frac{d}{dt} x(t) \longleftrightarrow j\omega X(\omega)$$

$$\int_{-\infty}^{t} x(t)dt \longleftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$$

推廣一下

Continuous time FT

$$x(t) \xrightarrow{F} X(\omega)$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) \xrightarrow{F^{-1}} x(t)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Discrete time FT

$$x[n] \xrightarrow{F} X(\Omega)$$

$$X(\Omega) = \sum_{-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

$$X(\Omega) \xrightarrow{F^{-1}} x[n]$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(\Omega) e^{j\Omega n} d\Omega$$

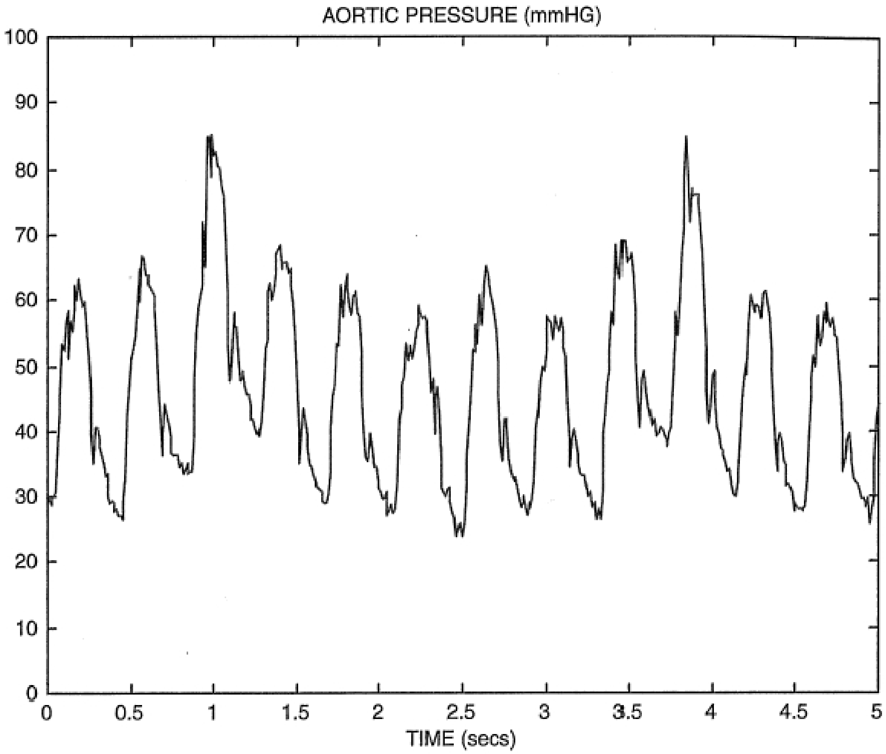
Discrete Fourier Transform (DFT)

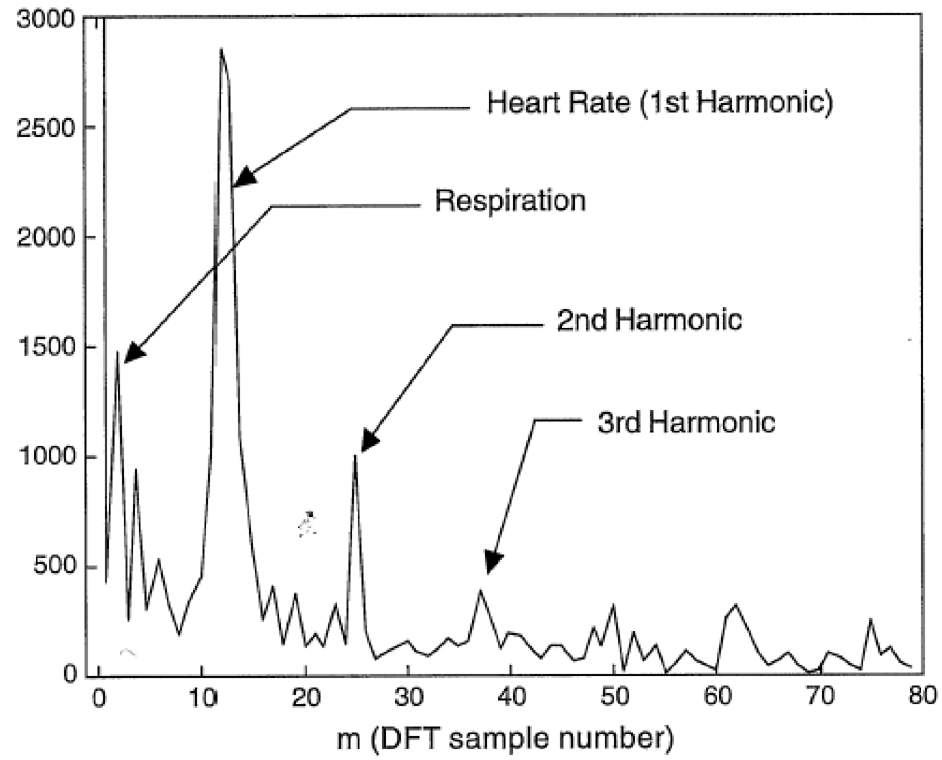
$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}, k = 0,1,\ldots, N-1$$

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk(2\pi/N)n}, n = 0,1,\ldots, N-1$$

- Both $x[n]$ and $X[k]$ are periodic with a cycle of N points.
- $\Omega = \frac{2\pi k}{N}$

Blood pressure waveform & its spectrum





Discrete Fourier transform

取樣

Sampling: the process from continuous to discrete

Sampling

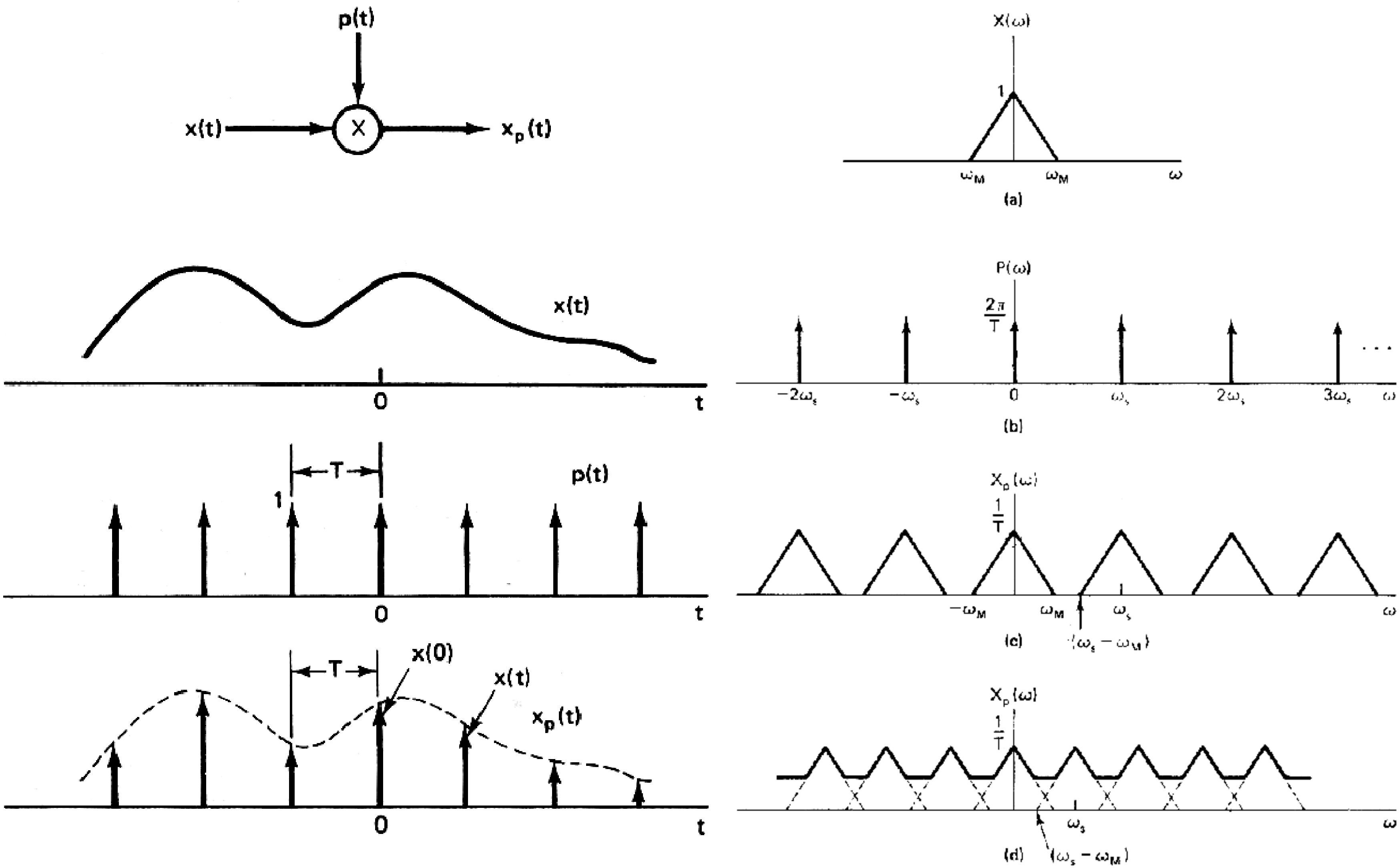
- Sampling process:
  - Convert $x(t)$ to a series of numbers $x[n]$
  - "n" is an integer
- Uniform sampling
  - $t = nT_s$, $f_s = 1/T_s$
  - $x[n] = x(nT_s) = x(n/f_s)$

Sampling

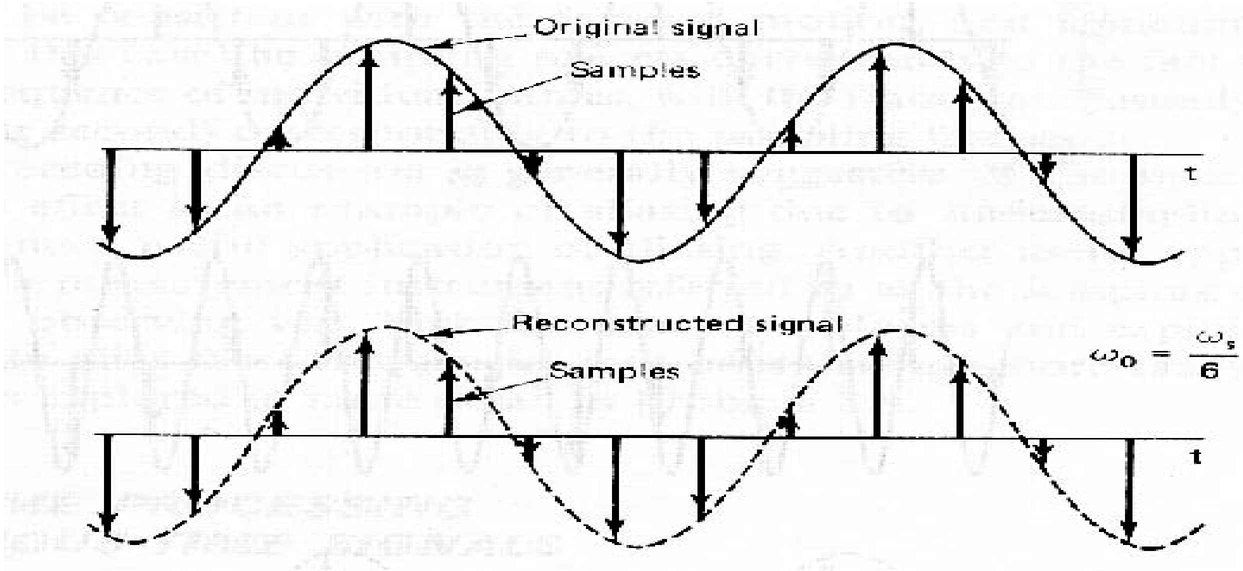
- 在時域取樣的過程，相當於在頻域的特定處（kf_s, k為整數）複製頻譜。

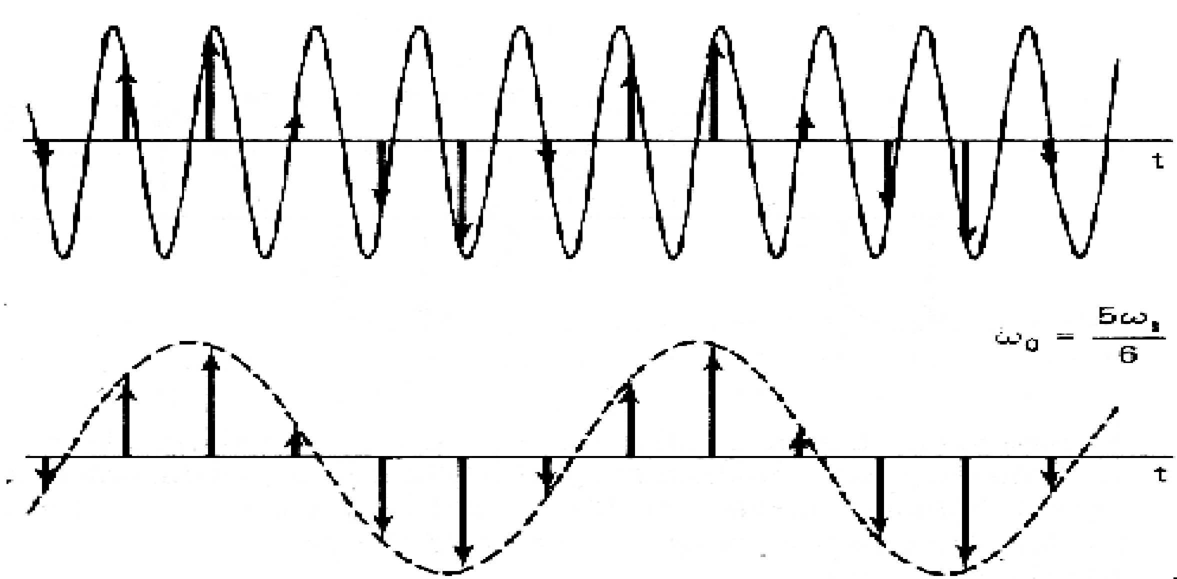
- Sampling theorem: $f_s \geqq 2 f_{max}$
- 若頻譜分佈太廣(或者說取樣頻率不夠高)，則會造成重疊失真(aliasing)。

Sampling



p(t)
x(t)
X
$x_p(t)$
$X(\omega)$
1
ω_M
ω_M
ω
(a)
x(t)
0
t
$P(\omega)$
$\frac{2\pi}{T}$
$-2\omega_s$
$-\omega_s$
0
ω_s
$2\omega_s$
$3\omega_s$
ω
(b)
T
p(t)
1
0
t
$X_p(\omega)$
$\frac{1}{T}$
$-\omega_M$
ω_M
ω_s
ω
(c)
$(\omega_s - \omega_M)$
T
x(0)
x(t)
$x_p(t)$
0
t
$X_p(\omega)$
$\frac{1}{T}$
ω_s
ω
(d)
$(\omega_s - \omega_M)$


Original signal
Samples
t
Reconstructed signal
Samples
$\omega_0 = \frac{\omega_s}{6}$
t


t
$\omega_0 = \frac{5\omega_s}{6}$
t

Aliasing!!

Aliasing

$x_1(t)$ 為低頻訊號 (200 Hz)，$x_2(t)$ 為高頻訊號 (1200 Hz)
Sampling frequency = 1000 Hz

$$x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_1[n] = \cos(400\pi \tfrac{n}{1000}) = \cos(0.4\pi n)$$

$$x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_2[n] = \cos(2400\pi \tfrac{n}{1000}) = \cos(2.4\pi n)$$

$$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$$

$$\Rightarrow x_2[n] = x_1[n]$$

濾波

Filter

濾波：filter

- 其實就是對訊號作加減乘除
- 改變參數即可獲得不同效果，運算彈性大
- 再來看一下一開始的例子吧

the following input–output equation

$$y[n] = \frac{1}{3}(x[n] + x[n+1] + x[n+2])$$

n	$n < -2$	-2	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	0	2	4	**6**	**4**	**2**	0	0
$y[n]$	0	$\frac{2}{3}$	2	4	$\frac{14}{3}$	**4**	2	$\frac{2}{3}$	0	0

$$y[0] = \frac{1}{3}(x[0] + x[1] + x[2])$$

$$y[1] = \frac{1}{3}(x[1] + x[2] + x[3])$$

Past, present, and future

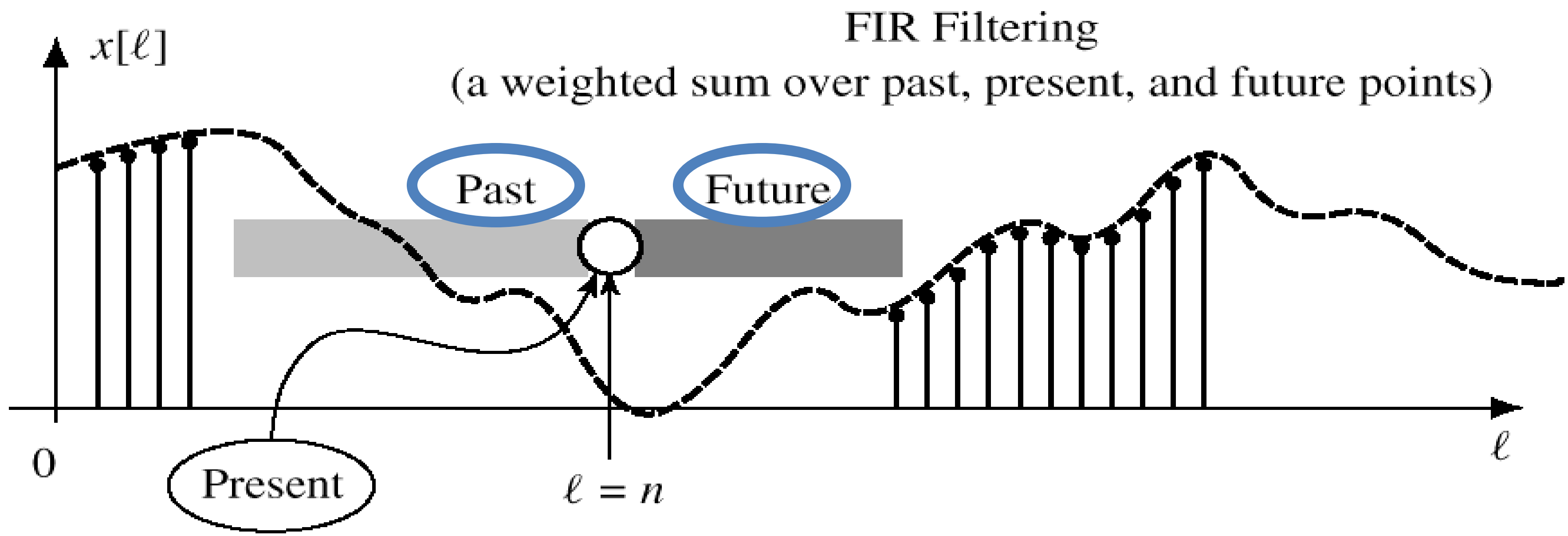


Figure 5.4 The running-average filter calculation at time index n uses values within a sliding window (shaded). Dark shading indicates the future ($\ell > n$); light shading, the past ($\ell < n$).

換個方向...

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

n	$n < -2$	-2	-1	0	1	2	3	4	5	6	7	$n > 7$
$x[n]$	0	0	0	2	4	**6**	**4**	**2**	0	0	0	0
$y[n]$	0	0	0	$\frac{2}{3}$	2	4	$\frac{14}{3}$	**4**	2	$\frac{2}{3}$	0	0

Real-time LTI system

Finite impulse response (FIR) filter

- 剛剛說的這些就是FIR filters

- From running average to weighting average
  - Define a real-time FIR by filter coefficients $\{b_k\}$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

  - Example: b_k = {3, -1, 2, 1}

$$y[n] = \sum_{k=0}^{3} b_k x[n-k]$$
$$= 3x[n] - x[n-1] + 2x[n-2] + x[n-3]$$

Time domain analysis

- 一些常用的統計數據：
  - Root-Mean-Square (RMS): the power of signal

$$RMS = \sqrt{\frac{\sum_{n=0}^{N-1} x^2[n]}{N}}$$

  - Average rectified value (ARV): estimation of the size of an alternating signal

$$ARV = \frac{\sum_{n=0}^{N-1} \left| x[n] \right|}{N}$$

How about IIR?

- Infinite impulse response filter
- 其實就是在頻域上做濾波

- 根據濾出的波段特性來分類濾波器
  - High pass filter: edge enhancement, remove DC
  - Low pass filter: attenuate high frequency noise
  - Band pass filter: band-selective/rejective

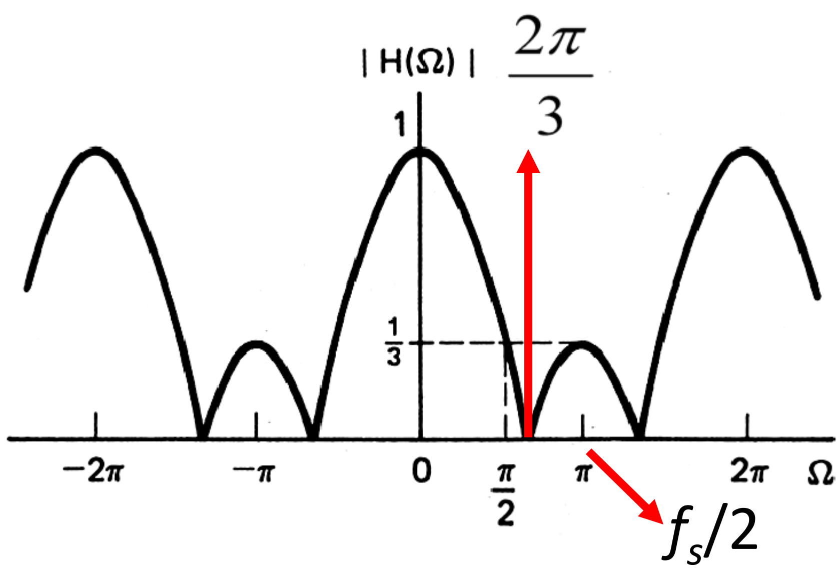
常見的訊號處理技巧

Ex1: Notch filter

就用剛剛的三點平均濾波器為例：

$$y(n) = \frac{1}{3}[x(n-2) + x(n-1) + x(n)]$$

Discrete-time FT



Also a low-pass filter!

For *fs* = 180 Hz, this filter can notch the 60 Hz artifacts.

Ex2: Signal averaging

- 生醫訊號 = 訊號 + 雜訊
- SNR還是太低怎麼辦？
- 來平均吧！
  - 訊號的部分視為不變
  - 通常雜訊被近似為 Gaussian noise with a zero mean

$$f_{noise}(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{x^2}{2\sigma^2})$$

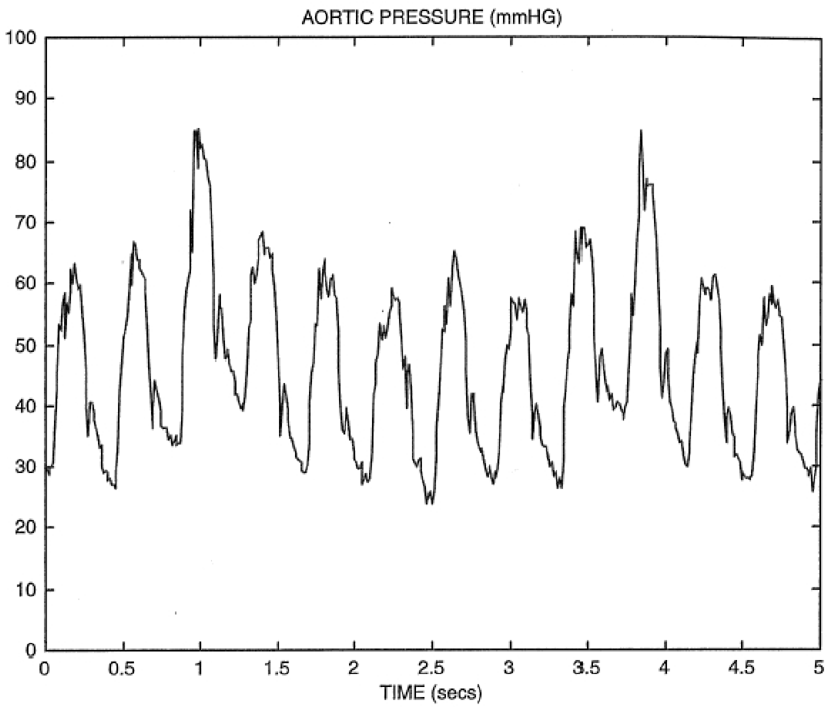
  - 多量幾次(N)平均起來

$$SNR = \frac{I(signal)}{noiselevel} \propto \sqrt{N}$$

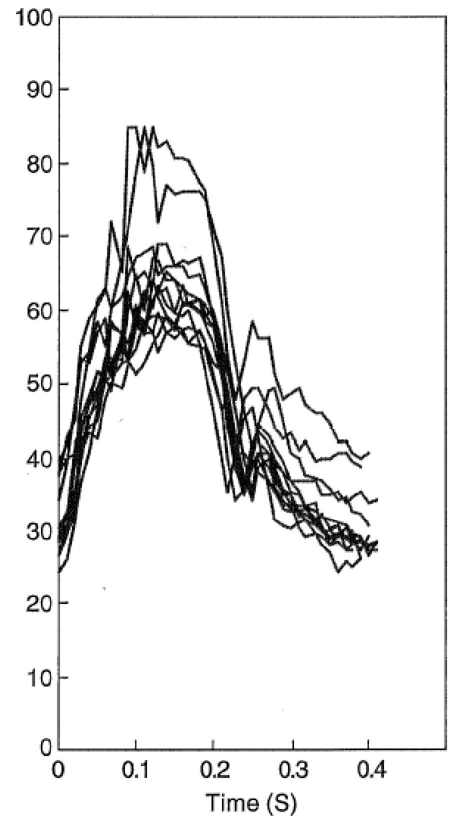
請注意使用前提…

- 可使用的場合：
  - Periodic signal: blood pressure
  - Repetitive stimulus: EEG
  - Mass and noisy data: Averaging in frequency domain

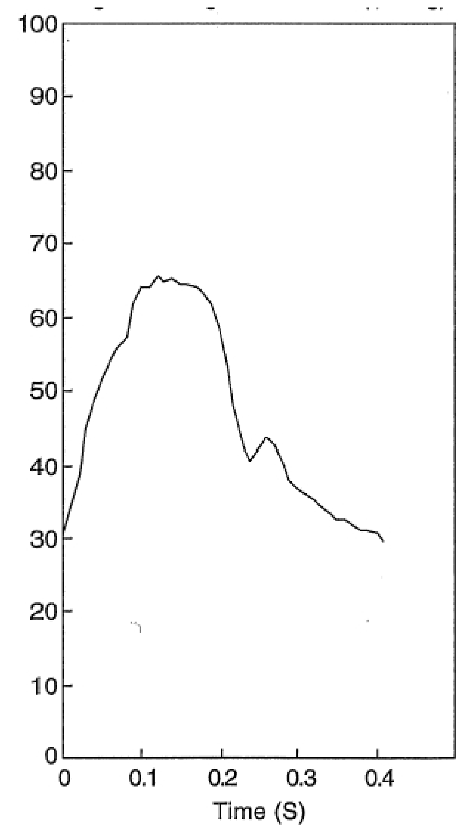
Averaging: periodic signal



Blood pressure signal
（具有明顯的週期性）

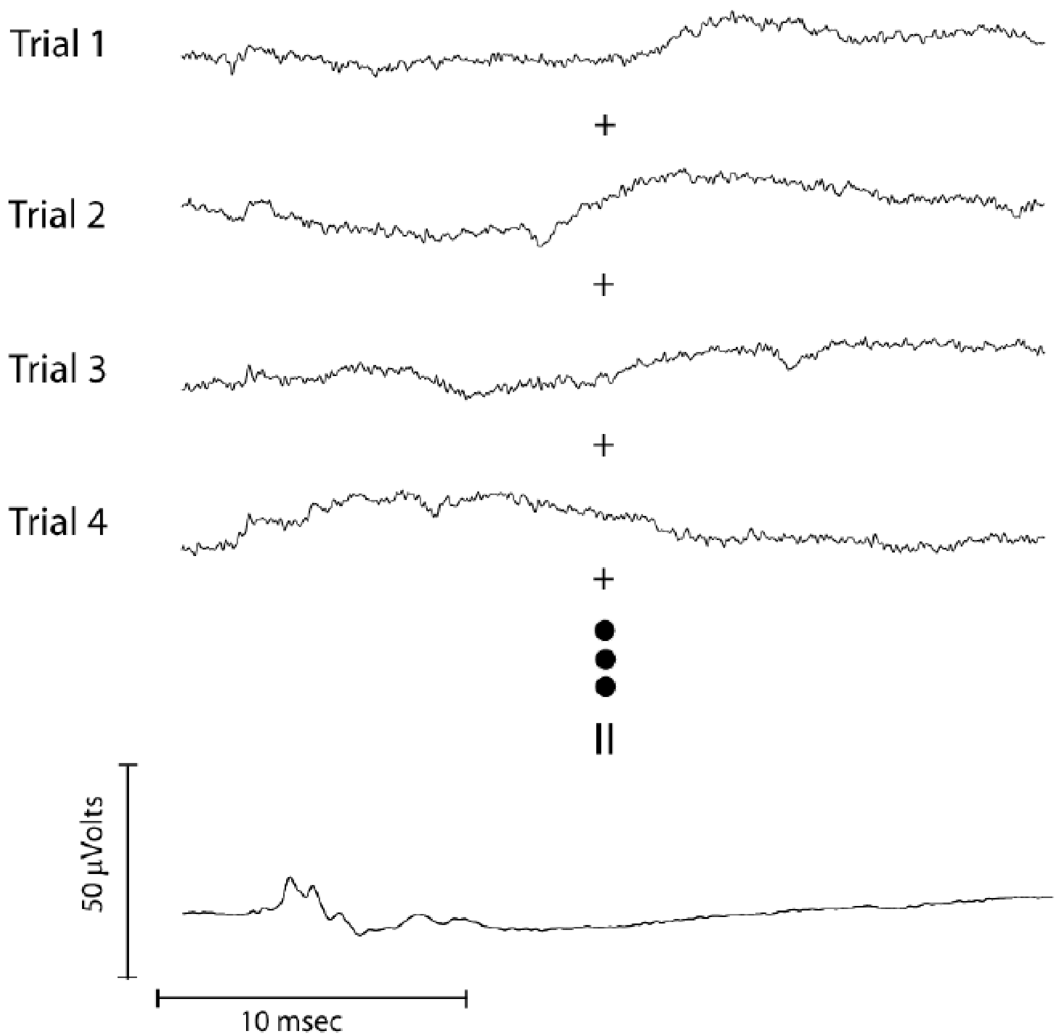


將每個週期切開疊在一起



平均波形

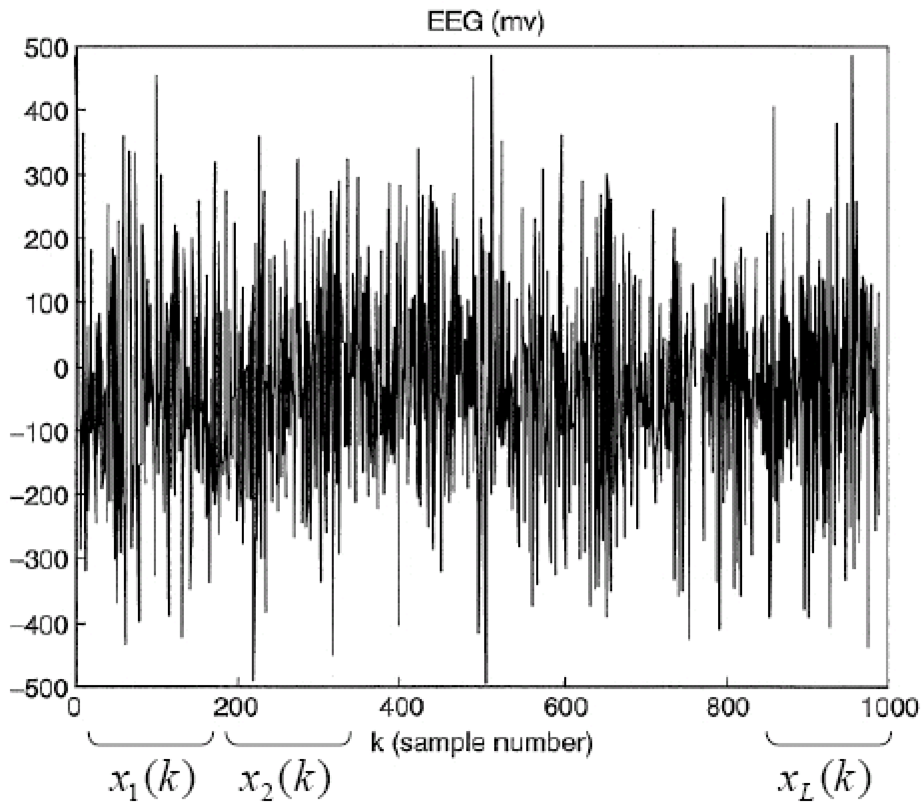
Averaging: repetitive stimulus

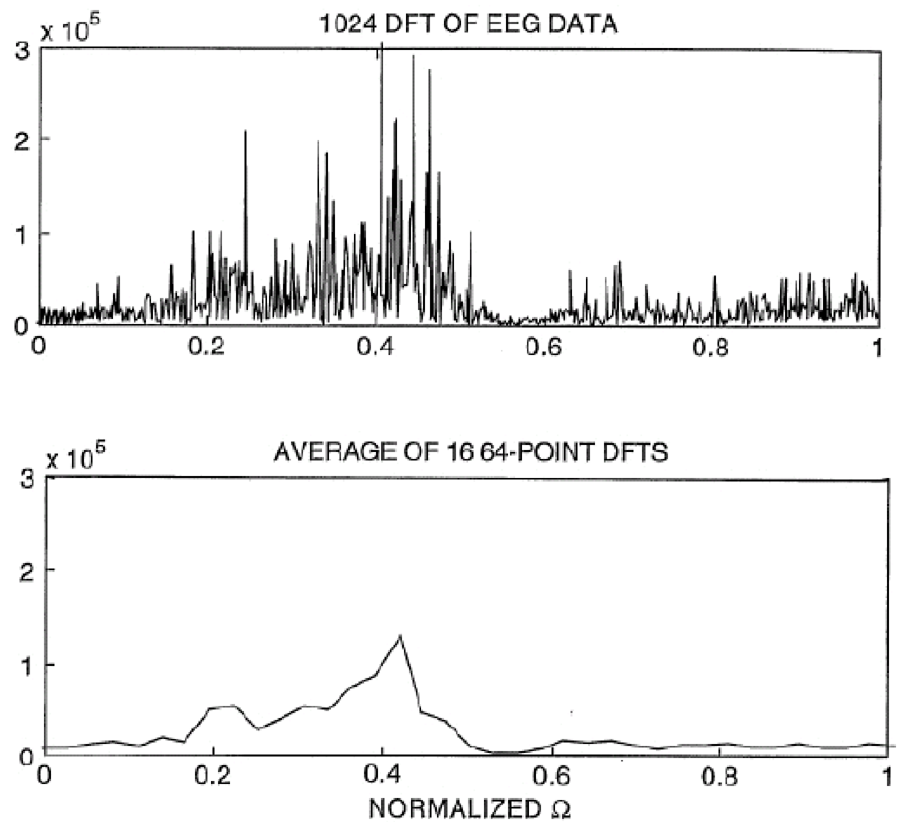


EEG experiment: Auditory evoked response (AER)

Averaged response of 1000 trials

Averaging: frequency domain





1024 points raw data

(f_s = 16 kHz)

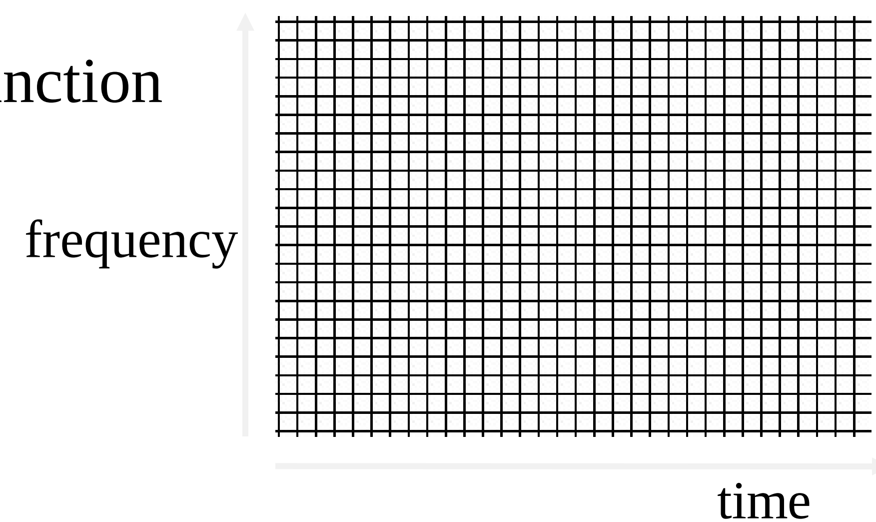
16-average result (trade-off?)

Ex3: Time-frequency analysis

- 頻譜特性隨時間變動
- 將收集的訊號適當分段
- Short-time Fourier transform (STFT)

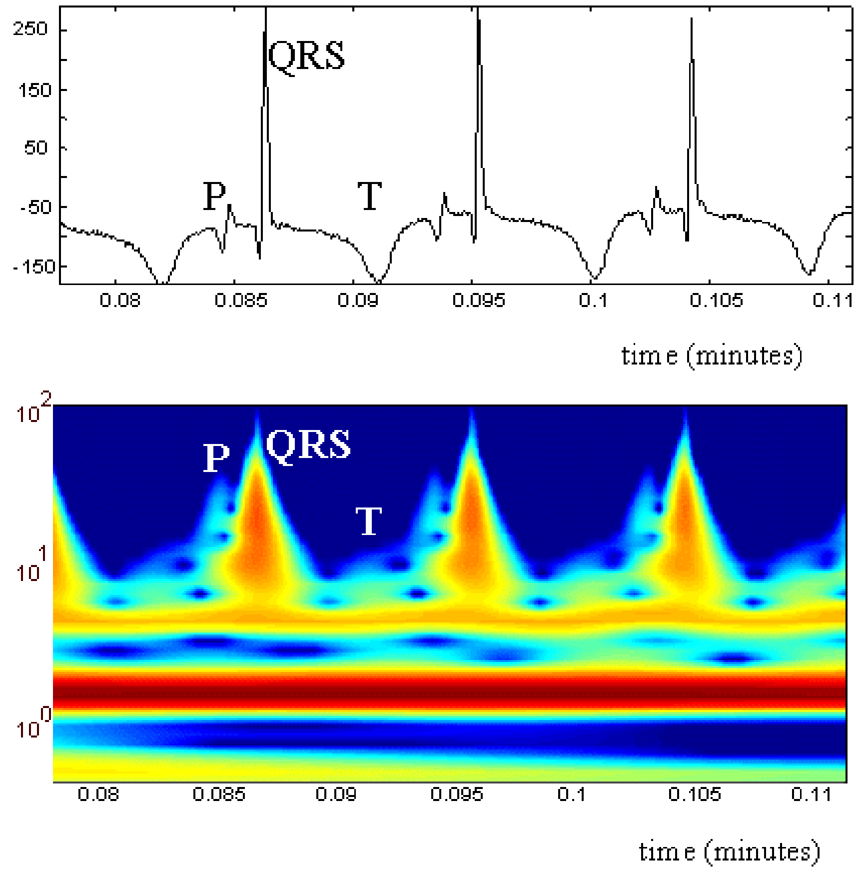
$$X(\omega, a) = \int_{-\infty}^{+\infty} x(t) g(t-a) \cdot e^{-j\omega t} dt$$

  - g(t): window function
  - a: time-shift of window function

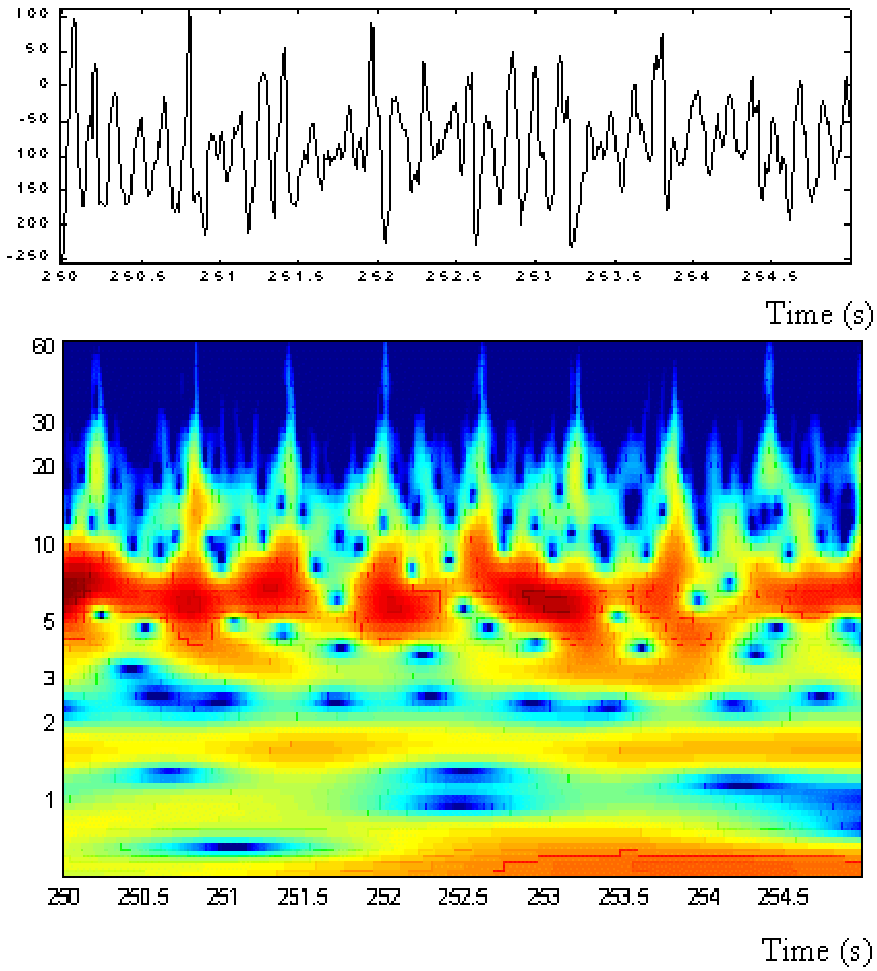


Time-frequency analysis of ECG

Normal



Ventricular Fibrillation



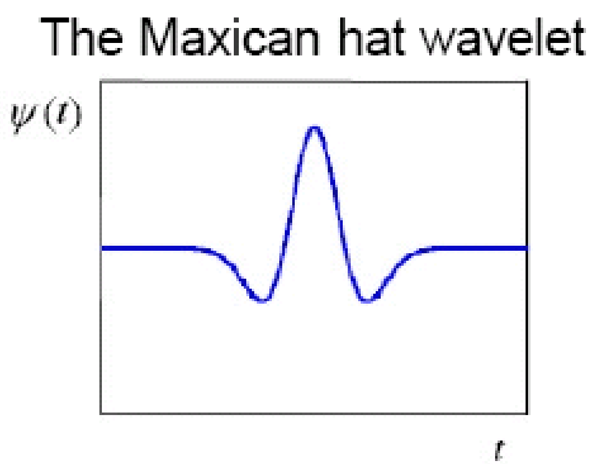
Short-time Fourier transform

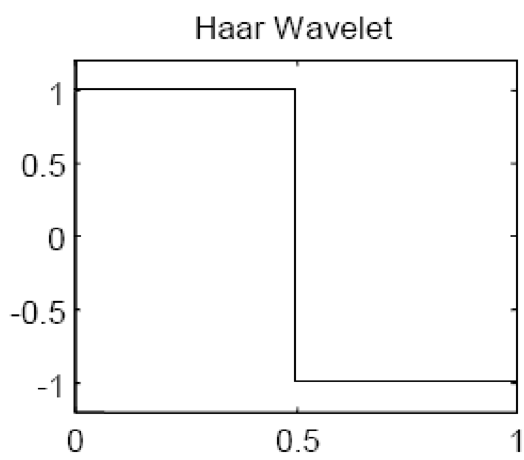
- Fixed window function
  - 萬一特徵長度比 window 還長？
- Sinusoidal components
  - 生醫訊號不見得都很平滑
  - 例: QRS-wave若用弦波分解，頻寬需求高

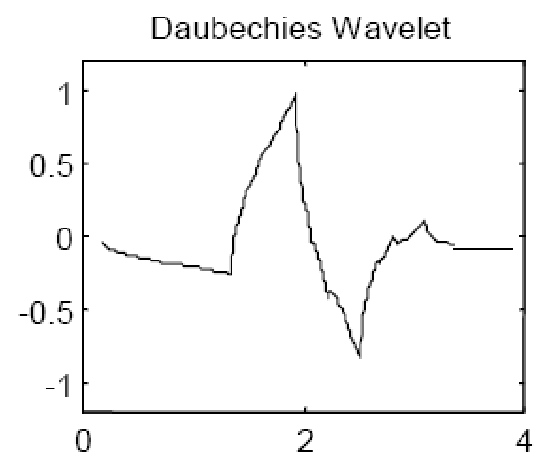
- 為解決以上問題
  - Wavelet transform

Wavelet transform

- Fourier transform: 將訊號分解成不同頻率的<u>弦波</u>
- 那可以是任意指定波形的組合嗎？
  - 可以，就是Wavelet transform !







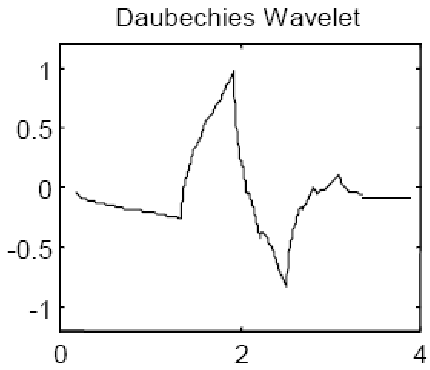
常見的三種 original Wavelets $\Psi(t)$

Wavelet transform

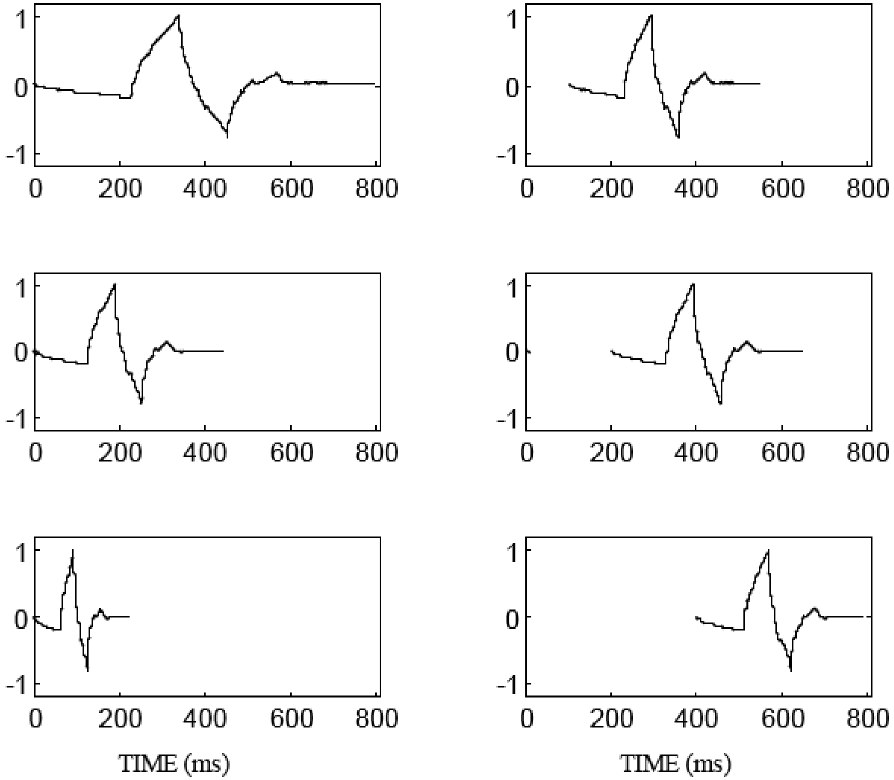
- Forward WT

$$H(a,s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{+\infty} x(t)\psi\left(\frac{t-a}{s}\right)dt$$

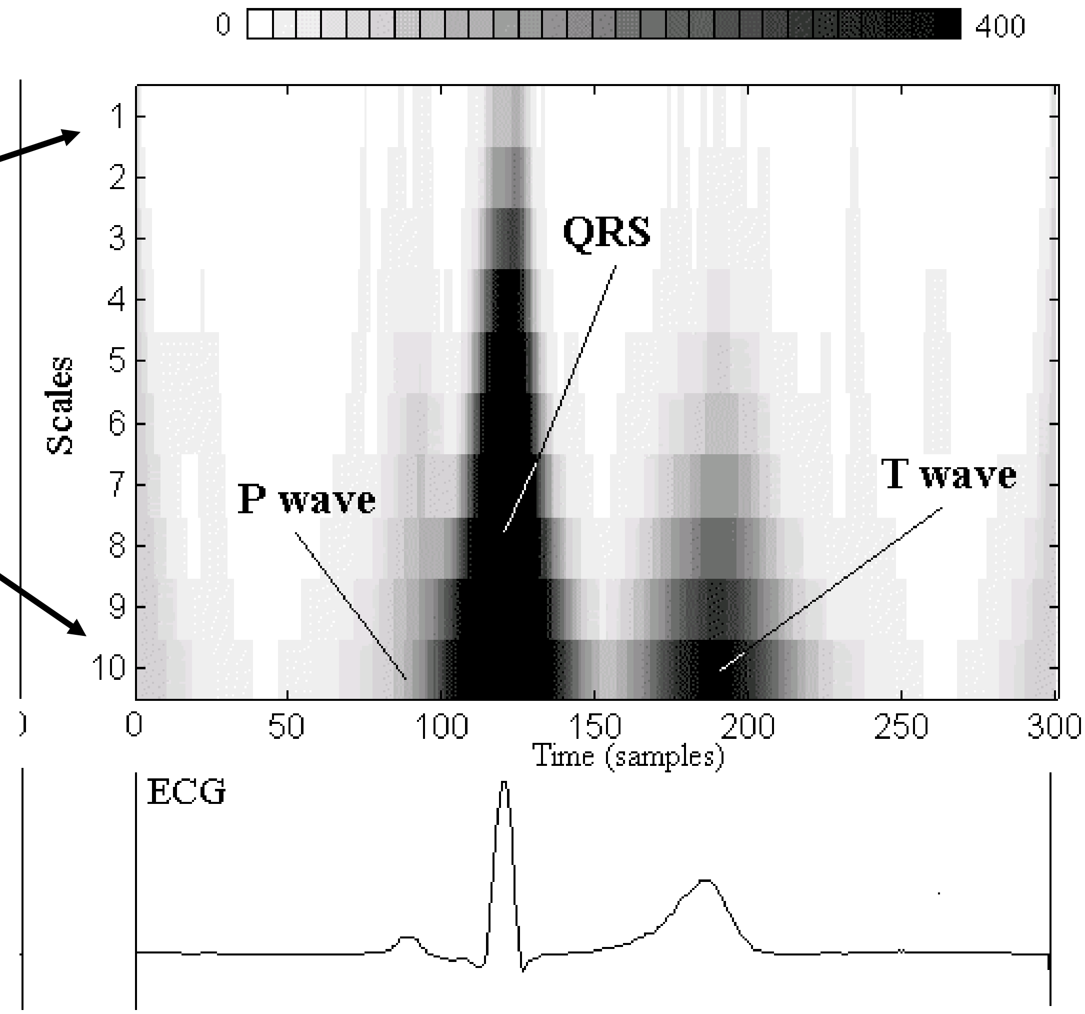
  - s : scale factor
  - a : position factor (shifting)
  - ψ(t): mother wavelet



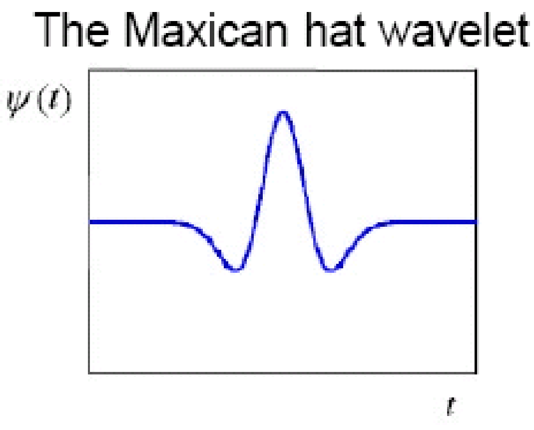
db2 ψ(t)



Example: ECG



High frequency
Low frequency
0
400
Scales
1
2
3
4
5
6
7
8
9
10
QRS
P wave
T wave
0
50
100
150
200
250
300
Time (samples)
ECG



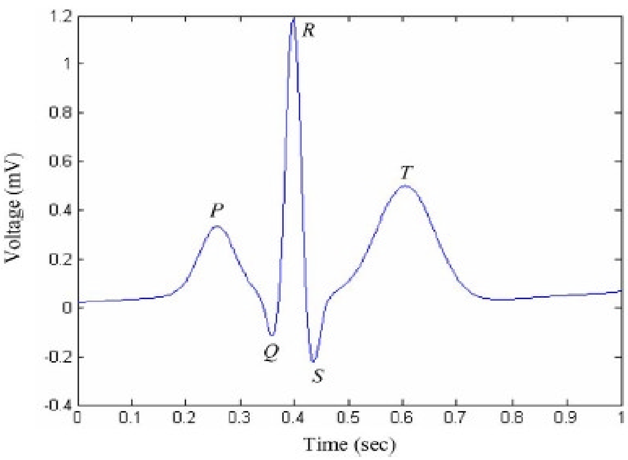
The Maxican hat wavelet
$\psi(t)$
t

Wavelet reconstruction

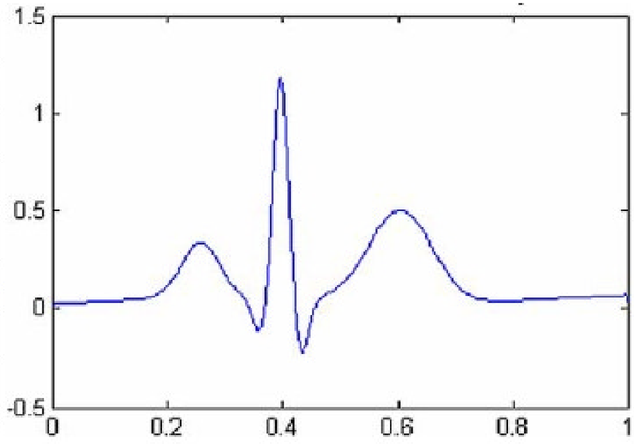
- Inverse WT

$$x(t) = \sum \sum H(a,s)\psi(a,s,t)$$

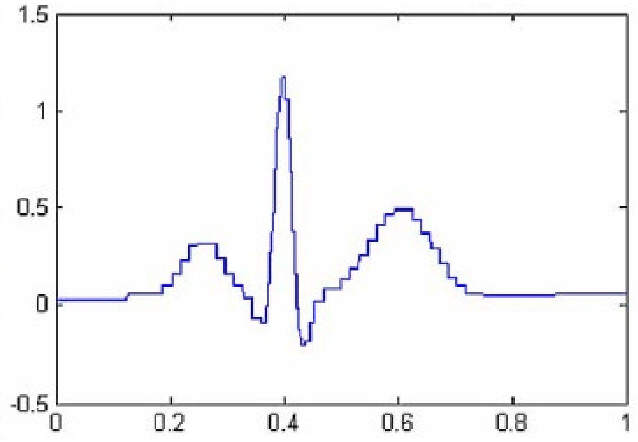
Original ECG signal



Reconstructed using db8



Reconstructed using Haar



Wavelet transform

- FT用不同頻率的弦波分解訊號，WT用不同比例的wavelet basis functions分解訊號。
  - Sinusoidal waves: infinite in time
  - Mother wavelets: finite
- 基本波形的多樣性，讓Wavelet 不受限於弦波的平滑變化！
  - 適合暫態的突發訊號

Wavelet transform

- 應用：filtering, denoising, compression, and feature extraction…

- 跟Fourier transform比起來，那個比較好？
  - 叉子和湯匙那個比較好用？
  - 看你想吃什麼…

Reference chapters:

Chapter 10: Biosignal processing, “Introduction to Biomedical Engineering”, John Enderle, Susan Blanchard, and Joseph Bronzio.

Oppenheim & Willsky, (1997), “Signals & Systems”, 2nd Ed., Prentice-Hall

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